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Yang–Baxterization of braid group representation associated with the seven-dimensional representation of G_2

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Abstract. The explicit expression of a solution that relates to the seven-dimensional representation of the Lie algebra G_2 of the quantum Yang-Baxter equation (QYBE) is obtained by applying the Yang–Baxterization procedure to the braid group representation (BGR). The result is consistent with an earlier one derived by a different method.

Recently the relation between the solutions of quantum Yang–Baxter equations (QYBE) and the representation of braid groups (BGR) has generated much interest [1–3]. Based on the quantum group (QG), a systematic method was formulated by Jimbo to generate the solutions of QYBE [4, 5] which are related to Lie algebras A_n, B_n, C_n and D_n in their fundamental representations. In [6], the BGR associated with the fundamental representation of G_2 was first obtained based on the general theory of the QG. The argument of Jimbo [5] has been followed to calculate the quantum R-matrix for G_2 in the fundamental seven-dimensional representation [7]. On the other hand, a prescription to give solutions of QYBE

$$\check{R}_{ij}^{ab}(x)\check{R}_{kf}^{jc}(xy)\check{R}_{de}^{ik}(y)=\check{R}_{ij}^{bc}(y)\check{R}_{dk}^{ai}(xy)\check{R}_{ef}^{kj}(x) \tag{1}$$

for a given BGR which satisfies

$$S_{ij}^{ab}S_{kf}^{jc}S_{de}^{ik}=S_{ij}^{bc}S_{dk}^{ai}S_{ef}^{kj} \tag{2}$$

has been discussed in [8]. We have called the procedure Yang–Baxterization, which is a generalization of the idea presented by Jones in [9]. The advantage of this approach is that it gives the explicit form of $\check{R}(x)$ directly from any given BGR S which provides all of the q-analogue projectors automatically. This prescription depends on the number of distinct eigenvalues of the considered BGR. On the basis of this approach, there have been much discussion about the 3-eigenvalue cases [8]. It has been shown that the BGR of G_2 has four distinct eigenvalues [7, 8], hence it provides the best example for checking our Yang–Baxterization procedure for the four-eigenvalue cases.

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The basic observation of the Yang–Baxterization comes from the loop-extension of the QG shown by [5, 10]. For the given co-multiplication Δ for a q-analogue of the universal enveloping algebra U_q, there are the basic equations [5, 10]:

$$[\check{R}(x), (\Pi\otimes\Pi)\Delta(U_q)]=0 \tag{3}$$

$$\check{R}(x)\{x\Pi(X_\theta^+)\otimes\Pi(k_\theta^{-1})+\Pi(k_\theta)\otimes\Pi(X_\theta^+)\} \\ =\{\Pi(X_\theta^+)\otimes\Pi(k_\theta^{-1})+x\Pi(k_\theta)\otimes\Pi(X_\theta^+)\}\check{R}(x) \tag{4}$$

where Π is the representation, x is the spectral parameter and θ stands for the largest root. It has been shown [6] that for a given direct product of Lie algebra representations

$$\Delta\otimes\Delta=\bigoplus_{i=1}^{m} E_i \tag{5}$$

and the decomposed irreducible spaces there are q-analogue projectors which are related to BGR S through

$$S=\sum_{i=1}^{m}\lambda_i P_i \tag{6}$$

where the λ_i's are the distict q-dependent eigenvalues. Obviously it holds that

$$\check{R}(x)=\sum_{i=1}^{m}\rho_i(x)P_i. \tag{7}$$

Since the solution of (3) and (4) satisfies (1), the consistence requirement leads to

$$SA=BS \tag{8}$$

with

$$\check{R}(x=0)=(\text{over all factor})S \tag{9}$$

and

$$A=\Pi(k_\theta)\otimes\Pi(X_\theta^+) \tag{10}$$

$$B=\Pi(X_\theta^+)\otimes\Pi(k_\theta^{-1}). \tag{11}$$

Now the question of Yang–Baxterization is how to construct $\check{R}$ from a given S.

Following from (8) and (6), we have in general

$$P_iAP_j=\left(\frac{\lambda_j}{\lambda_i}\right)P_iBP_j. \tag{12}$$

Here the ordering of λ_i has to be chosen so that $\check{R}$ is constructed satisfying (1). The P_i's are known for a given S, hence we can calculate P_iAP_j if a special ordering of λ_i exists such that the following hold

$$P_iAP_i=P_iBP_i \tag{13}$$

$$P_{i+1}AP_i=\left(\frac{\lambda_i}{\lambda_{i+1}}\right)P_{i+1}BP_i \tag{14}$$

$$P_iAP_{i+1}=\left(\frac{\lambda_{i+1}}{\lambda_i}\right)P_iBP_{i+1} \tag{15}$$

and all other $P_iAP_j=0$. Then we have

$$\frac{\rho_i(x)}{\rho_{i+1}(x)}=\frac{x+\lambda_i/\lambda_{i+1}}{1+x(\lambda_i/\lambda_{i+1})}. \tag{16}$$

After lengthy but elementary calculation for the four-eigenvalue case, we arrive at

$$\check{R}(x)=A(x)S^2+B(x)S+C(x)I+D(x)S^{-1} \tag{17}$$

where

$$A(x)=\frac{(\lambda_4\lambda_2-\lambda_1\lambda_3)x(x-1)}{(\lambda_2\lambda_3\lambda_4)(\lambda_4-\lambda_1)}$$

$$B(x)=-\frac{x-1}{\lambda_4}-\frac{[(\lambda_2+\lambda_3)(\lambda_2\lambda_4-\lambda_1\lambda_3)+\lambda_2\lambda_4^2-\lambda_1^2\lambda_3]x(x-1)}{\lambda_2\lambda_3\lambda_4(\lambda_4-\lambda_1)}$$

$$\begin{aligned}C(x)=\frac{1}{(\lambda_2\lambda_3\lambda_4)(\lambda_4-\lambda_1)}\{&[(\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4)(\lambda_2\lambda_4-\lambda_1\lambda_3)\\&+\lambda_2\lambda_3(\lambda_4^2-\lambda_1^2)+\lambda_1\lambda_4(\lambda_3\lambda_4-\lambda_1\lambda_2)]x^2\\&+[\lambda_3^2\lambda_4(\lambda_1+\lambda_2)-\lambda_1\lambda_2^2(\lambda_3+\lambda_4)]x\}\end{aligned} \tag{18}$$

$$D(x)=x(x-1)\lambda_1\left(x+\frac{\lambda_3-\lambda_2}{\lambda_4-\lambda_1}\right).$$

Now the procedure of constructing $\check{R}$ can be listed as follows:

(i) first find S for a given Lie algebraic structure (5);

(ii) by using

$$P_i=\prod_{j=1,i\neq j}^{m}\frac{S-\lambda_j}{\lambda_i-\lambda_j} \tag{19}$$

the project P_i can be found;

(iii) check (13)-(15) and fix the ordering of the eigenvalues λ_i;

(iv) substitute S and the ordered λ_i into (17) and (18); we find $\check{R}(x)$ which satisfies (1).

We emphasize that for the 'exotic solutions' we can take (17) as the starting point and directly check the derived $\check{R}(x)$ to satisfy (1) even for some models without projectors [11].

Now we apply the above procedure to the G_2 case in order to obtain $\check{R}(x)$ related to the seven-dimensional representation of the algebra. First we list the BGR S associated with the representation of G_2 [6, 12]:

$$S=\text{block diag}[A_6, A_5, \ldots, A_1, A_0, A_1, \ldots, A_5, A_6] \tag{20}$$

where

$$A_6=(u)$$

$$A_5=\begin{pmatrix}0 & p_1\\ p_1 & w_1\end{pmatrix}$$

$$A_4=\begin{pmatrix}0 & 0 & p_1\\ 0 & u & 0\\ p_1 & 0 & w_1\end{pmatrix}$$

$$A_3=\begin{pmatrix}0 & 0 & 0 & p_3\\ 0 & 0 & p_2 & q_1\\ 0 & p_2 & w_2 & q_2\\ p_3 & q_1 & q_2 & w_3\end{pmatrix}$$

$$A_2=\begin{pmatrix}0 & 0 & 0 & 0 & p_5\\ 0 & 0 & 0 & p_4 & q_3\\ 0 & 0 & u & 0 & 0\\ 0 & p_4 & 0 & w_4 & q_4\\ p_5 & q_3 & 0 & q_4 & w_5\end{pmatrix} \tag{21}$$

$$A_1=\begin{pmatrix}0 & 0 & 0 & 0 & 0 & p_8\\ 0 & 0 & 0 & 0 & p_7 & 0\\ 0 & 0 & 0 & p_6 & 0 & q_5\\ 0 & 0 & p_6 & w_6 & 0 & q_6\\ 0 & p_7 & 0 & 0 & w_7 & 0\\ p_8 & 0 & q_5 & q_6 & 0 & w_8\end{pmatrix}$$

$$A_0=\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & p_{11}\\ 0 & 0 & 0 & 0 & 0 & p_{10} & q_{12}\\ 0 & 0 & 0 & 0 & p_9 & q_9 & q_{13}\\ 0 & 0 & 0 & 1 & q_7 & q_8 & q_{11}\\ 0 & 0 & p_9 & q_7 & w_9 & q_{10} & q_{14}\\ 0 & p_{10} & q_9 & q_8 & q_{10} & w_{10} & q_{15}\\ p_{11} & q_{12} & q_{13} & q_{11} & q_{14} & q_{15} & w_{11}\end{pmatrix}$$

with

$$p_1=p_7=u^{1/2} \qquad p_2=p_5=p_8=u^{-1/2} \qquad p_3=p_4=p_6=1 \qquad p_9=p_{10}=p_{11}=u^{-1}$$

$$w_1=u-1 \qquad w_2=u-u^{-2} \qquad w_3=w_1(1+u^{-3}) \qquad w_4=w_6=u-u^{-1}$$

$$w_5=w_8=w_1(u^{-2}+u^{-3}+1) \qquad w_7=w_1 \qquad w_9=w_1(1-u^{-2})$$

$$w_{10}=w_1(1-u^{-5}) \qquad w_{11}=w_9(1+u^{-2})^2 \qquad q_1=[u^{-1}(1+u^{-1})]^{1/2}w_1$$

$q_2 = q_1(-u^{-3/2})$ $\quad q_3 = q_5 = q_1$ $\quad q_4 = q_6 = -q_1 u^{-1}$ $\quad q_7 = w_6 u^{-1/2}$

$q_8 = -u^{-2} w_6$ $\quad q_9 = u^{-1/2} w_2$ $\quad q_{10} = u^{-7/2} w_1$ $\quad q_{11} = u^{-1/2} w_3$

$q_{12} = u^{-3/2} w_1$ $\quad q_{13} = -u^{-3} w_1$ $\quad q_{14} = -u^{-1}(1 + u^{-1} + u^{-3}) w_1$

$q_{15} = -u^{-3/2} q_{14}.$

The eigenvalue equations for the above matrices are (λ are the eigenvalues):

$$\begin{aligned}
A_6:&\quad (\lambda - u) = 0\\
A_5:&\quad (\lambda - u)(\lambda + 1) = 0\\
A_4:&\quad (\lambda - u)^2(\lambda + 1) = 0\\
A_3:&\quad (\lambda - u)^2(\lambda + 1)(\lambda + u^{-3}) = 0\\
A_2:&\quad (\lambda - u)^3(\lambda + 1)(\lambda + u^{-3}) = 0\\
A_1:&\quad (\lambda - u)^3(\lambda + 1)^2(\lambda + u^{-3}) = 0\\
A_0:&\quad (\lambda - u)^3(\lambda + 1)^2(\lambda + u^{-3})(\lambda - u^{-6}) = 0.
\end{aligned} \tag{22}$$

The recursion relation is

$$(S + u^{-3})(S - u)(S + 1)(S - u^{-6}) = 0. \tag{23}$$

In order to compare with [7], we first diagonalize S by introducing the orthogonal transformation and then picking up each submatrix corresponding to the distinct eigenvalues. The result is as follows, where the overall factor $u^{1/2}$ has been omitted.

For $\lambda = u$

$$L_u = \text{block diag}[L_1, L_2, \ldots, L_6, L_7, L_6, \ldots, L_1] \tag{24}$$

where

$$L_1 = 1$$

$$L_2 = [\boldsymbol{\alpha}_{21}, 0],\ \boldsymbol{\alpha}_{21} = \frac{1}{\sqrt{1+u}}\begin{pmatrix} 1 \\ u^{1/2} \end{pmatrix}$$

$$L_3 = [\boldsymbol{\alpha}_{31}, \boldsymbol{\alpha}_{32}, 0],\ \boldsymbol{\alpha}_{31} = \frac{1}{\sqrt{1+u}}\begin{pmatrix} 1 \\ 0 \\ u^{1/2} \end{pmatrix} \qquad \boldsymbol{\alpha}_{32} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$L_4 = [\boldsymbol{\alpha}_{41}, \boldsymbol{\alpha}_{42}, 0, 0],\ \boldsymbol{\alpha}_{41} = \frac{1}{\sqrt{c}}\begin{pmatrix} u^{-1/2} \\ u^{1/2}/a \\ 1/a \\ u^{1/2} \end{pmatrix} \qquad \boldsymbol{\alpha}_{42} = \frac{1}{\sqrt{b}}\begin{pmatrix} -u^{-1/2} \\ u^{-3/2}a \\ ua \\ -u^{1/2} \end{pmatrix}$$

with $a=\sqrt{1+u}$, $b=u^3+u^2+u+u^{-1}+u^{-2}+u^{-3}$, $c=u+1+u^{-1}$,

$$L_5=[\boldsymbol{\alpha}_{51},\boldsymbol{\alpha}_{52},\boldsymbol{\alpha}_{53},0,0]$$

$$\boldsymbol{\alpha}_{51}=\frac{u^{1/2}}{\sqrt{c}}\begin{pmatrix}(ua)^{-1}\\ u^{-1/2}\\ 0\\ u^{-1/2}\\ u^{1/2}a^{-1}\end{pmatrix}\qquad \boldsymbol{\alpha}_{52}=\frac{1}{\sqrt{b}}\begin{pmatrix}-(ua)^{-1}\\ u^{-3/2}\\ 0\\ u^{3/2}\\ -au^{1/2}\end{pmatrix}\qquad \boldsymbol{\alpha}_{53}=\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}$$

$$L_6=[\boldsymbol{\alpha}_{61},\boldsymbol{\alpha}_{62},\boldsymbol{\alpha}_{63},0,0,0]$$

$$\boldsymbol{\alpha}_{61}=\frac{1}{\sqrt{u+u^{-1}}}\begin{pmatrix}0\\0\\u^{-1/2}\\u^{1/2}\\0\\0\end{pmatrix}\qquad \boldsymbol{\alpha}_{62}=\frac{1}{a}\begin{pmatrix}0\\1\\0\\0\\u^{1/2}\\0\end{pmatrix}\qquad \boldsymbol{\alpha}_{63}=\frac{1}{\sqrt{d}}\begin{pmatrix}u+u^{-1}\\ 0\\ (u^2+u)^{1/2}(u-1\\ -au^{-1/2}(u-1)\\ 0\\ u^{3/2}(u+u^{-1})\end{pmatrix}$$

with $d=u^5+u^4+u^3+u^2+u^1+1+u^{-1}+u^{-2}$;

$$L_7=[\boldsymbol{\alpha}_{71},\boldsymbol{\alpha}_{72},\boldsymbol{\alpha}_{73},0,0,0,0]$$

$$\boldsymbol{\alpha}_{71}=\frac{1}{\sqrt{g}}\begin{pmatrix}0\\0\\u^{-1}\\u^{-1/2}(u+1)\\u\\0\\0\end{pmatrix}\qquad \boldsymbol{\alpha}_{72}=\frac{1}{\sqrt{h(u+u^{-1})}}\begin{pmatrix}1+u^{-2}\\ u^{-1/2}(u+u^{-1})\\ u-u^{-1}\\ u^{-3/2}(u^2-1)(u-1)\\ u^{-1}-u\\ u^{1/2}(u+u^{-1})\\ u^2+1\end{pmatrix}$$

$$\boldsymbol{\alpha}_{73}=\frac{1}{\sqrt{cl}}\begin{pmatrix}-u^{-1}\\ u^{-3/2}c\\ u(c+u^{-2})\\ -u^{1/2}(c+u^{-2})\\ c+u^{-2}\\ u^{3/2}c\\ -uc\end{pmatrix}$$

with

$$g=u^2+u+2+u^{-1}+u^{-2}\ h=u^3+2u^2+u+u^{-1}+2u^{-2}+u^{-3},$$

$$l=u^4+au^3+2u^2+u+2+3u^{-1}+4u^{-2}+3u^{-3}+u^{-4}.$$

For $\lambda=-1$

$$\bar{L}_{-1}=\text{block diag}[\bar{L}_1,\bar{L}_2,\ldots,\bar{L}_6,\bar{L}_7,\bar{L}_6,\ldots,\bar{L}_1]\tag{25}$$

where

$$\bar{L}_1 = 0$$

$$\bar{L}_2 = [0, \boldsymbol{\beta}_{22}] \qquad \boldsymbol{\beta}_{22} = \frac{1}{a}\begin{pmatrix} u^{1/2} \\ -1 \end{pmatrix}$$

$$\bar{L}_3 = [0, 0, \boldsymbol{\beta}_{33}] \qquad \boldsymbol{\beta}_{33} = \frac{1}{a}\begin{pmatrix} u^{1/2} \\ 0 \\ -1 \end{pmatrix}$$

$$\bar{L}_4 = [0, 0, \boldsymbol{\beta}_{43}, 0] \qquad \boldsymbol{\beta}_{43} = \frac{1}{\sqrt{a_1}}\begin{pmatrix} -a \\ -u \\ u^{-1/2} \\ a \end{pmatrix}$$

with $a_1 = u^2 + 2u + 2 + u^{-1}$;

$$\bar{L}_5 = [0, 0, 0, \boldsymbol{\beta}_{54}, 0] \qquad \boldsymbol{\beta}_{54} = \frac{1}{\sqrt{a_1}}\begin{pmatrix} -1 \\ -u^{1/2}a \\ 0 \\ au^{-1/2} \\ au^{1/2} \end{pmatrix}$$

$$\bar{L}_6 = [0, 0, 0, \boldsymbol{\beta}_{64}, \boldsymbol{\beta}_{65}, 0] \qquad \boldsymbol{\beta}_{64} = \frac{1}{\sqrt{a_1}}\begin{pmatrix} -1 \\ 0 \\ -au^{1/2} \\ au^{-1/2} \\ 0 \\ u^{1/2} \end{pmatrix} \qquad \boldsymbol{\beta}_{65} = \frac{1}{a}\begin{pmatrix} 0 \\ u^{1/2} \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

$$\bar{L}_7 = [0, 0, 0, \boldsymbol{\beta}_{74}, \boldsymbol{\beta}_{75}, 0, 0]$$

$$\boldsymbol{\beta}_{74} = \frac{1}{\sqrt{b_1}}\begin{pmatrix} -u^{-1/2} \\ -1 \\ -u^{-1/2}(u+1) \\ -u^{-1}(u^2-1) \\ u^{-1/2}(u+1) \\ 1 \\ u^{1/2} \end{pmatrix} \qquad \boldsymbol{\beta}_{75} = \frac{1}{\sqrt{b_2}}\begin{pmatrix} u^{-1/2} \\ -u^{-1} \\ -u^{-1/2}(u^2-1) \\ u^{-1}(u^2-1) \\ -u^{-3/2}(u^2-1) \\ u \\ -u^{1/2} \end{pmatrix}$$

with $b_1 = u^2 + 3u + 4 + 3u^{-1} + u^{-2}$, $b_2 = u^3 + 2u^2 - 2 + 2u^{-2} + u^{-3}$.

For $\lambda = -u^{-3}$

$$\tilde{L}_{-u^{-3}} = \text{block diag}[\tilde{L}_1, \tilde{L}_2, \ldots, \tilde{L}_6, \tilde{L}_7, \tilde{L}_6, \ldots, \tilde{L}_1] \tag{26}$$

where

$$\tilde{L}_1 = 0,\ \tilde{L}_2 = 0,\ \tilde{L}_3 = 0$$

$$\tilde{L}_4 = [0, 0, 0, \gamma_{44}],\ \gamma_{44} = \frac{1}{\sqrt{b}} \begin{pmatrix} -u^{3/2} \\ au^{1/2} \\ -au^{-1} \\ u^{-3/2} \end{pmatrix}$$

$$\tilde{L}_5 = [0, 0, 0, 0, \gamma_{55}] \qquad \gamma_{55} = \frac{1}{\sqrt{b}} \begin{pmatrix} -au \\ u^{1/2} \\ 0 \\ -u^{-1/2} \\ au^{-3/2} \end{pmatrix}$$

$$\tilde{L}_6 = [0, 0, 0, 0, 0, \gamma_{66}] \qquad \gamma_{66} = \frac{1}{\sqrt{b}} \begin{pmatrix} -au \\ 0 \\ u^{1/2} \\ -u^{-1/2} \\ 0 \\ au^{-3/2} \end{pmatrix}$$

$$\tilde{L}_7 = [0, 0, 0, 0, 0, \gamma_{76}, 0] \qquad \gamma_{76} = \frac{1}{\sqrt{b}} \begin{pmatrix} -u \\ -u^{3/2} \\ 1 \\ -u^{-1/2}(u-1) \\ -1 \\ u^{-3/2} \\ u^{-1} \end{pmatrix}.$$

For $\lambda = u^{-6}$

$$\tilde{\tilde{L}}_{u^{-6}} = \text{block diag}[\tilde{\tilde{L}}_1, \tilde{\tilde{L}}_2, \ldots, \tilde{\tilde{L}}_6, \tilde{\tilde{L}}_7, \tilde{\tilde{L}}_6, \ldots, \tilde{\tilde{L}}_1] \tag{27}$$

where

$$\tilde{\tilde{L}}_1 = 0 \qquad \tilde{\tilde{L}}_2 = 0 \qquad \tilde{\tilde{L}}_3 = 0 \qquad \tilde{\tilde{L}}_4 = 0 \qquad \tilde{\tilde{L}}_5 = 0 \qquad \tilde{\tilde{L}}_6 = 0$$

$$\tilde{\tilde{L}}_7 = [0, 0, 0, 0, 0, 0, \gamma_{77}] \qquad \gamma_{77} = \frac{1}{\sqrt{c_1}} \begin{pmatrix} u^{5/2} \\ -u^2 \\ u^{1/2} \\ -1 \\ -u^{1/2} \\ -u^2 \\ -u^{5/2} \end{pmatrix}$$

with $c_1 = u^5 + u^4 + u + 1 + u^{-1} + u^{-4} + u^{-5}$.

Noting that the labelling set for G_2 is (3, 2, 1, 0, −1, −2, −3), the above results coincide with those in [7].

Finally we set the ordering of λ_i such that $P_iAP_{i+1}\neq 0$. It is found that

$$\lambda_1=-u^{-3} \qquad \lambda_2=u \qquad \lambda_3=-1 \qquad \lambda_4=u^{-6}. \tag{28}$$

Substituting these eigenvalues and (21) into (17) and (18), we derive the solution of (1):

$$\check{R}(x)=\text{block diag}[R^1, R^2, \ldots, R^6, R^7, R^6, \ldots, R^2, R^1] \tag{29}$$

where R^i is an $i\times i$ symmetric matrix. We list all the non-zero elements as follows:

$$R^1_{11}=(x-u)(x-u^6)(1-u^{-4}x)$$
$$R^2_{11}=x(u-1)(1-u^{-4}x)(u^6-x)$$
$$R^2_{12}=u^{-7/2}(x-1)(x-u^4)(u^6-x)$$
$$R^2_{22}=(1-u)(x-u^6)(1-u^{-4}x)$$
$$R^3_{11}=R^2_{11}$$
$$R^3_{13}=R^2_{12}$$
$$R^3_{22}=R^1_{11}$$
$$R^3_{33}=R^2_{22}$$
$$R^4_{11}=u^{-4}x(u^2-1)(x-u^6)[x(1-u+u^2)-u^3]$$
$$R^4_{12}=u^{-2}(u+1)^{1/2}x(x-1)(u^6-x)$$
$$R^4_{13}=u^{-7/2}(u+1)^{1/2}(u-1)x(1-x)(u^6-x)$$
$$R^4_{14}=u^{-3}(1-x)(x-u^3)(x-u^6)$$
$$R^4_{22}=u^{-4}x(1-u)(u^6-x)[x(1+u+u^2)-u(1+u+u^3)]$$
$$R^4_{23}=u^{-5/2}(1-x)(u^6-x)(u^2-x)$$
$$R^4_{24}=u^{-1}(u-1)(x-1)(1+u)^{1/2}(x-u^6)$$
$$R^4_{33}=u^{-4}(u-1)(x-u^6)[x(1+u^2+u^3)-u^2(1+u+u^2)]$$
$$R^4_{34}=u^{-5/2}(u+1)^{1/2}(x-1)(u-1)(u^6-x)$$
$$R^4_{44}=u^{-4}(u-1)(x-u^6)(x-u+u^2-u^3)$$
$$R^5_{11}=u^{-4}x(u-1)(x-u^6)[x(1+u^2+u^3)-u^2(1+u+u^2)]$$
$$R^5_{12}=u^{-5/2}x(u+1)^{1/2}(x-1)(u-1)(u^6-x)$$
$$R^5_{14}=-u^{-7/2}x(u+1)^{1/2}(1-x)(1-u)(u^6-x)$$
$$R^5_{15}=u^{-5/2}(1-x)(u^6-x)(u^2-x)$$
$$R^5_{22}=u^{-4}x(u^2-1)(x-u^6)(x-u+u^2-u^3)$$
$$R^5_{24}=u^{-3}(1-x)(x-u^3)(x-u^6)$$
$$R^5_{25}=u^{-1}(u+1)^{1/2}(1-x)(u-1)(u^6-x)$$
$$R^5_{33}=R^1_{11}$$
$$R^5_{44}=u^{-4}(u^2-1)(x-u^6)[x(1-u+u^2)-u^3]$$

$$R^5_{45}=u^{-2}(u+1)^{1/2}(x-1)(u-1)(u^6-x)$$
$$R^5_{55}=u^{-4}(u-1)(x-u^6)[x(1+u+u^2)-u(1+u+u^3)]$$
$$R^6_{11}=R^5_{24}$$
$$R^6_{13}=R^5_{12}$$
$$R^6_{14}=R^5_{14}$$
$$R^6_{16}=R^5_{15}$$
$$R^6_{22}=R^2_{11}$$
$$R^6_{25}=R^2_{12}$$
$$R^6_{33}=R^5_{22}$$
$$R^6_{34}=R^4_{14}$$
$$R^6_{36}=R^5_{25}$$
$$R^6_{44}=R^5_{44}$$
$$R^6_{46}=R^5_{45}$$
$$R^6_{55}=R^2_{22}$$
$$R^6_{66}=R^5_{55}$$
$$R^7_{11}=u^{-4}x^2(u-1)(u^4-1)[u^2(u^4+1)-x(u^2+1)]$$
$$R^7_{12}=u^{-3/2}x(x-1)(u-1)[x(1+u+u^3)-u^5(1+u+u^2)]$$
$$R^7_{13}=u^{-3}x(x-1)(u-1)[u^5(1+u+u^2)-x(1+u+u^3)]$$
$$R^7_{15}=u^{-1}x(u-1)(x-1)(u^2-x)$$
$$R^7_{14}=u^{-7/2}x(x-1)(u^2-1)[x(1-u+u^2)-u^5]$$
$$R^7_{16}=u^{-5/2}x(u-1)(x-1)(x-u^2)$$
$$R^7_{17}=u^{-2}(1-x)(u^2-x)(u^5-x)$$
$$R^7_{22}=u^{-4}x(u-1)^2[u^7(1+u+u^2)+xu^4(1+u^2)-x^2(1+u+u^2+u^3+u^4)]$$
$$R^7_{23}=u^{-1/2}x(u-1)(1-x)(u^2-x)$$
$$R^7_{24}=u^{-2}x(u^2-1)(x-1)(u^5-u^4+u^3-x)$$
$$R^7_{25}=u^{-7/2}x(1-x)(u-1)[u^7+u^6+u^4-x(u^2+u+1)]$$
$$R^7_{26}=u^{-2}(1-x)(u^2-x)(u^5-x)$$
$$R^7_{27}=u^{5/2}(u-1)(x-1)(x-u^2)$$
$$R^7_{33}=u^{-4}x(u-1)^2(1+u)[u^4(1+u^2+u^4)-x(1+u^2)-x^2]$$
$$R^7_{34}=u^{-7/2}x(u^2-1)(1-x)(u^5-u^4+u^3-x)$$
$$R^7_{35}=u^{-2}(u^2-x)(1-x)(u^5-x)$$
$$R^7_{36}=u^{-3/2}(1-x)(u-1)[u^5(u^2+u+1)-(u^3+u+1)x]$$
$$R^7_{37}=u(x-1)(u-1)(u^2-x)$$
$$R^7_{44}=u^{-4}(u^3-x)[ux^2+x(1-2u+u^3-2u^4+u^5-2u^7+u^8)+u^7]$$
$$R^7_{45}=u^{-1/2}(1-x)(u-1)(u+1)(u^5-u^2x+ux-x)$$
$$R^7_{46}=u^{-2}(x-1)(u^2-1)(u^5-u^2x+ux-x)$$
$$R^7_{47}=u^{-1/2}(1-x)(u^2-1)(u^5-u^4+u^3-x)$$

$$R^7_{55} = u^{-4}(u-1)^2(1+u)[u^8 + xu^4(1+u^2) - x^2(1+u^2+u^4)]$$
$$R^7_{56} = u^{1/2}(u-1)(1-x)(u^2-x)$$
$$R^7_{57} = u^{-2}(x-1)(u-1)[u^4(1+u^2+u^3) - x(u^2+u+1)]$$
$$R^7_{66} = u^{-4}(u-1)^2[u^5(1+u+u^2+u^3+u^4) - xu^3(1+u^2) - x^2(1+u+u^2)]$$
$$R^7_{67} = u^{-7/2}(x-1)(u-1)[x(1+u+u^2) - u^4(1+u+u^3)]$$
$$R^7_{77} = u^{-4}(u-1)(u^4-1)[u^4(u^2+1) - x(u^4+1)].$$

This $\check{R}(x)$-matrix provides Boltzmann weights of an 175-vertex model [7]. A direct check also confirmed that this $\check{R}(x)$ satisfies (1), hence we complete the derivation of $\check{R}(x)$ for G_2 from the point of view of the Yang–Baxterization.

To conclude the paper we remark that the starting point of the method employed here is the braid relation. Although it is parallel to the method of [7] for the case treated here, it is applicable to the 'exotic solution'. We will treat the 'exotic solution' of the G_2 case in a further publication.

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References

[1] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)
[2] Yang C N and Ge M L (eds) 1989 *Braid Group, Knot Theory and Statistical Mechanics* (Singapore: World Scientific)
[3] Wadati M, Deguchi T and Akutsu Y 1989 *Phys. Rev.* **180** 247
[4] Jimbo M 1989 *Braid Group, Knot Theory and Statistical Mechanics* ed C N Young and M L Ge (Singapore: World Scientific) p 111-34
[5] Jimbo M 1986 *Commun. Math. Phys.* **102** 537
[6] Yu N Reshetikhin 1990 *Preprint* LOMI E-4-87, E-17-87
[7] Kuniba A 1990 *J. Phys. A: Math. Gen.* **23** 1349
[8] Ge M L, Wu Y S and Xue K 1990 Explicit trigonometric Yang–Baxterization *Preprint* ITP-SB-90-92
[9] Jones V 1985 *Commun. Math. Phys.* **125** 63
[10] Jimbo M 1985 *Lett. Math. Phys.* **10** 63
[11] Ge M L and Xue K 1990 New Solutions of Braid Group Representations Associated with the Yang–Baxter Equation *Preprint* ITP-SB-90-20
[12] Ge M L, Li Y Q and Xue K 1990 *Int. J. Mod. Phys.* **5** 1975